Robustifying Learnability

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Abstract

In recent years, the learnability of rational expectations equilibria (REE) and determinacy of economic structures have rightfully joined the usual performance criteria among the sought-after goals of policy design. Some contributions to the literature, including Bullard and Mitra [2001] and Evans and Honkapohja [2002], have made significant headway in establishing certain features of monetary policy rules that facilitate learning. However a treatment of policy design for learnability in worlds where agents have potentially misspecified their learning models has yet to surface. This paper provides such a treatment. We begin with the notion that because the profession has yet to settle on a consensus model of the economy, it is unreasonable to expect private agents to have collective rational expectations. We assume that agents have only an approximate understanding of the workings of the economy and that their learning the reduced forms of the economy is subject to potentially destabilizing perturbations. The issue is then whether a central bank can design policy to account for perturbations and still assure the learnability of the model. We provide two examples one of which—the canonical New Keynesian business cycle model—serves as a test case. For different parameterizations of a given policy rule, we use structured singular value analysis (from robust control theory) to find the largest ranges of misspecifications that can be tolerated in a learning model without compromising convergence to an REE.

In addition, we study the cost, in terms of performance in the steady state of a central bank that acts to robustify learnability on the transition path to REE. (Note: This paper contains full-color graphics)

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1 Introduction

It is now widely accepted that policy rules—and in particular, monetary policy rules—should not be chosen solely on the basis of their performance in a given model of the economy. There is simply too much uncertainty about the true structure of the economy to warrant taking the risk of so narrow a criterion for selection. Rather, policy should be robustified; that is, designed to operate "well" in a wide range of environments. There has been substantial progress in a relatively short period of time in the literature on robustifying policy. One strand of the literature examines the performance of rules given the presence of measurement errors in either model parameters or unobserved state variables.1 A second strand focuses on comparing rules in rival models to see if their performance spanned reasonable sets of alternative worlds.2 A third considers robustifying policy against unknown alternative worlds, usually by invoking robust control methods.3

At roughly the same time the literature on robustifying policy was unfolding, another literature was being developed on learning in macroeconomics. The learning literature takes a step back from rational expectations and asks whether the choices of uninformed private agents could be expected to converge on a rational expectations equilibrium (REE) as the outcome of a process of learning. Important early papers in this literature include Bray [1982], Bray and Savin [1986] and Marcet and Sargent [1989]. Evans and Honkapohja summarize some of their many contributions to this literature in their book [2001]. A cornerstone of this research has been the E-stability principle. The principle says that if an ordinary differential equation (ODE) that represents the mapping of agents beliefs (the perceived law of motion) onto the true dynamics of the economy (the actual law of motion) converges on a fixed point, the economy with its associated learning rule is stable. Moreover, the convergence of the meta-time ODE on a fixed point implies local convergence in real time of a recursive process—specifically, recursive least squares (RLS)—on rational expectations

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1 Brainard [1967] is the seminal reference. Among the many, more recent references in this large literature are Sack [1999], Orphanides et al. [2000], Soderstrom [2002] and Ehrmann and Smets [2003].

2 See, e.g., Levin et al. [1999] and [2003].

3 Hansen and Sargent [2008] and [2003], Tetlow and von zur Muehlen [2004], von zur Muehlen (1982) and Coenen [2007]. These strands of the robustness literature are named in the text in chronological order but the three methods should be seen as complementary rather than substitutes.
equilibrium (REE). The close association of E-stability with learnability has meant that in much of the literature, the terms have been used more-or-less interchangeably. However, recent papers, e.g., Giannitsarou [2005] and Evans et al.[2005], show that there is some loss of generality in doing so: there is no assurance that agents use RLS for learning, and Giannitsarou [2005] provides an example in which the E-stability principle holds—and RLS would converge on REE—but the stochastic gradient learning would not. This is a pertinent observation because stochastic gradient learning differs from RLS only in relatively small ways; moreover, the example provides for us the impetus for a test case in a way that will become clear later on.

In monetary economies, the central bank is part of the data generating process; that is, it is part of the actual law of motion. As such, the question arises: could monetary policy assist or hinder private agents in their learning of the REE? If so, what could be done to ensure that agents’ learning has the best prospects possible for convergence? In other words, how could one robustify learnability? In this paper, we address this question by taking a strand of the robust policy literature and marry it with the E-stability literature.

The common features of the robust policy literature include, first, that it is the government that does not understand the true structure of the economy, and second, that the government’s ignorance will not vanish simply with the collection of more data. By contrast, in the learning literature it is usually the private sector that is assumed not to have the information necessary to form rational expectations, but this situation has at least the prospect of being alleviated with the passage of time and the collection of more data.

With the profession as a whole unable to agree on a workhorse model of the economy, it seems unreasonable to us to expect private agents to have rational expectations at all points in time. It seems reasonable to us to posit that agents have an approximate understanding of the workings of the economy and that they are on a transition path toward learning the

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4 "The connection between E-stability and the convergence of least squares learning turns out to be quite general, applying to a very wide range of models. This is a great advantage since E-stability conditions are often easy to work out, while the technical analysis of convergence of econometric learning is substantially more involved." Evans and Honkapohja [2001, p.31.] This generality has been called into question in the more recent literature.

5 The concept of "truth" is a slippery one in learning models. In some sense, the truth is jointly determined by the deep structural parameters of the economy and what people believe them to be. Only in steady state, and then only under some conditions, will this be solely a function of deep parameters and not of beliefs. Nevertheless, in this paper, when we refer to a "true model" or "truth" we mean the REE upon which successful learning eventually converges.
true structure. And as Evans and McGough [2007] show, designing policy as if the process of learning has already been completed can result in indeterminacy. It follows that for a central bank, the task of doing what it can to assure REE obtains is logically prior to the task of maximizing social welfare once the REE comes about. Thus, in this paper we retain the assumption of adaptive learning that is common with most contributions to this literature.

In this paper, we consider two issues. The first is how a policy maker might choose policy to maximize the set of possible worlds in which private agents are capable of learning the REE. The second is an assessment of the welfare cost of assuring learnability in terms of forgone stability in equilibrium. Or, put differently, we measure the welfare cost of learnability insurance. Each of these questions is important. If an ill-chosen policy rule leads to explosiveness or indeterminacy, the performance criteria used in most analyses of monetary policy rules is likely to be a moot point. At the same time, excessive concern for learnability will imply costs in terms of forgone welfare.

Ours is not the first paper to consider the issue of choosing monetary policies for their ability to deliver determinacy and expectational stability. Bernanke and Woodford [1997] argue that inflation-forecast-based (IFB) policy rules—that is, rules that feed back on forecasts of future output or inflation—can lead to indeterminacy in linear rational expectations (LRE) models. Clarida, Gali and Gertler [1998] show that violation of the so-called Taylor principle in the context of a forecast-based rule may have been the source of the inflation of the 1970s. Bullard and Mitra [2007] in an important paper show that higher persistence in instrument setting—meaning a large coefficient on the lagged instrument in a Taylor-type rule—can facilitate determinacy in the same class of models. Evans and Honkapohja [2003] survey some of this literature in the context of monetary policy design; Evans and McGough [2007] study monetary policy rules that are designed subject to a learnability constraint. Of particular pertinence to what we study here are the papers of Giannitsarou [2005] and Evans et al. [2005] because they consider departures from RLS that can break the link from the E-stability principle to learnability in the real-time (or recursive) sense of that term.

Each of these papers makes an important contribution to the literature, but all are special

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6 Levin et al. [2003] and Batini and Pearlman [2002] study the robustness properties of different types of inflation-forecast based rules for their stability and determinacy properties.
cases within broader sets of policy choices. In this paper, we follow a somewhat different approach and consider the design of policies to maximize E-stability of the economy.

The remainder of the paper is organized as follows. The second section lays out the theory, beginning with a review of the literature on least-squares learnability and determinacy, and following with methods from the robust control literature. Section 3 introduces the models with which we work, beginning with the case of the very simple Cagan model of money demand in hyperinflations and then moving on to the New Keynesian business cycle (NKB) model. For the NKB model, we study the design of time-invariant simple monetary policy rules to robustify learnability of three types: a lagged-information rule, a contemporaneous information rule and a forecast-based policy rule. We then cover the insurance cost of robustifying learnability. A fourth section provides an intriguing example of where a robustified inflation-forecast-based policy rule could avoid non-convergence of recursive learning by agents using stochastic gradient learning. A fifth and final section sums up and concludes.

2 Theoretical overview

2.1 Expectational equilibrium under adaptive learning

The theory of determinacy and E-stability in linear models is well established. Consequently, our review here is just long enough to allow us to contrast RLS learning with SG learning, later in the paper. Toward this end, we assume here that the model is determinate and focus on E-stability. The conditions for determinacy and some intermediate steps on E-stability are in the appendix.

Let us begin with the following linear rational expectations model:

\[ y_t = A + M E^*_t y_{t+1} + N y_{t-1} + P v_t, \]  

where \( y_t \) is a vector of \( n \) endogenous variables, including, possibly, policy instruments, and \( v_t \) comprises all \( m \) exogenous variables. Equation (1) is general in that in REE both non-predetermined variables, \( E^*_t y_{t+1} \), and predetermined variables, \( y_{t-1} \), are represented. The asterisk superscript on the expectations operator indicates that outside of REE we will
entertain the possibility that agents use RLS to form expectations. Let us assume that \( v_t \) is observable and follows a first-order stochastic process,

\[
v_t = \rho v_{t-1} + \epsilon_t,
\]

where \( \epsilon_t \) is an iid white noise process. We consider the minimum state variable (MSV) representation of the REE, advanced by McCallum [1983]. The appendix shows that the MSV solution will satisfy the mapping from perceived law of motion (PLM)—just a recursively estimated least squares regression of the MSV representation of the economy—to the actual law of motion (ALM):

\[
y_t = A + M(I + b)a + (N + Mb^2)y_{t-1} + (M(bc + c\rho) + P)\epsilon_t
\]

where \( a, b \) and \( c \) are RLS regression coefficients. Note that equation (2) includes both the regression coefficients of the PLM as well as the parameters of the underlying structural model. The MSV solution will satisfy the mapping from PLM to ALM, and thereby assure us that the model is E-stable, if certain eigenvalue conditions for the following matrix differential equations are satisfied:

\[
\begin{align*}
\frac{da}{d\tau} &= [A + M(I + b)]a - a, \\
\frac{db}{d\tau} &= Mb^2 + N - b, \\
\frac{dc}{d\tau} &= M(bc + c\rho) + P - c.
\end{align*}
\]

The important points to take from equations (3), called a T-mapping in the literature, are that the conditions are generally multivariate in nature—meaning that the coefficients constraining the intercept term, \( a \), can be conflated with those of the slope term, \( b \), and that the coefficients of both the PLM regression parameters and the model’s structural parameters come into play.

To date, applications in the literature have been largely confined to very simple, small-scale models where these problems are rare. In the kind of medium- to large-scale models that policy institutions use, these issues cannot be safely ignored. Without taking away anything from the important contributions of Bullard and Mitra [2007] and Evans and Honkapohja
[2003], the choice of monetary policy rules must not only consider how they foster learnability in a given model but whether they do so for the broader class of learning models within which the true learning model might be found. Similarly, taking as given the true model, the initial beliefs of private agents can affect learnability both through the inclusion and exclusion of states to the PLM and through the initial values attached to parameters. In the context of the above example, values of $a$, $b$, and $c$ that are initially "too far" from equilibrium can block convergence. The choice of a particular policy can shrink or expand the range of values for $a$, $b$, and $c$ that is consistent with E-stability. These are our concerns in this paper: how can a policy maker deal with uncertainty in agents’ learning mechanisms in the choice of his or her policy rule and thereby maximize the prospect that the economy will converge successfully on a rational expectations equilibrium? For this, we work with perturbations to the T-mapping described above or systems like it. We take this up in the next subsection.

2.2 Structured robust control

Unlike least-squares learning, structured robust control is not very familiar to economists. True, there is the new text by Hansen and Sargent [2008], but while the tools covered in that book are related to what we do here, they are not the same. Accordingly, in this subsection, we devote some space to introducing the pertinent methods from robust control theory.

To date, the literature has usually taken as given, first, that agents use least-squares learning to adapt their perceptions of the true economic structure, and second, that they know the correct linear or linearized form of the REE solution. Both of these assumptions can be questioned. We will not directly take up the issue of the linearity of agents decision rules in this paper, except to note that to the extent that a departure from linearity can be taken as a perturbation to agents’ reference model, nonlinearities do fall within the purview of the methods employed herein. We retain the assumption that the central bank uses a linear model with least-squares learning on the part of agents as its reference model, but assume

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7 In fact, in this simple example, the intercept coefficient, $a$, turns out to be irrelevant for the determination of learnability, although this result is not general.

8 Evans and Honkapohia [2001] survey variations on least-squares learning, including under- and over-parameterized learning models and discounted (or constant-gain) least squares learning. Still, in macroeconomics it is generally either least-squares learning or constant gain learning that is assumed. An exception is Marcet and Nicolini [2003], and Giannitsarou [2005].
that the central bank entertains doubts about its specification of the process of learning. Accordingly, we analyze these doubts using only a minimum amount of structure, drawing on structured model uncertainty, a subset of the robust control literature.

The approach taken here differs in two ways from what most economists think of when they think of robust control. First, robust control in monetary economies has generally been concerned with robust performance: the central bank is assumed to retain as its objective the minimization of a conventional loss function, even as it entertains the prospect of model misspecification. We consider a central bank worried about how large agents' learning errors can be before learning fails to drive the economy towards eventual equilibrium; thus the central bank is concerned with robust stability. (We will, however, have a look at the "insurance premium" that must be paid for robust stability later on.) Second, robust control usually regards model uncertainty as unstructured, in the sense that the misspecification is not ascribed to any particular feature of the model. Instead the fear of misspecification is usually represented by a set of additional shock variables under the control of an "evil agent" bent on causing harm.9 Structured model uncertainty shares with its unstructured sibling a concern for uncertainty in the sense of Knight—meaning that the uncertainty is assumed to be nonparametric, but carries the tag "structured" for two reasons. First, structured robust control associates the uncertainty with particular parts of the model. Onatski and Williams [2003] argue persuasively that central banks know what aspects of their models are questionable, either because of specification decisions made early in the process of formulating their models, or because of inherent weaknesses in econometric estimators given a formulation. And second, the perturbations themselves can be assigned a variety of structures, including time-series specifications.10

Boiled down to its essence, the five steps to designing policies subject to the constraint that agents must adapt to those policies and a learning model that may be misspecified are:

1. Write down a structural model of the economy and compute the conditions necessary

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9 The unstructured robust control problem in monetary economics is typically set up as a zero-sum game played by a maximizing central bank and a minimizing "malevolent nature". See, in particular, Sargent [1999], Giannoni [2002], Hansen and Sargent [2008, especially Part III], Tetlow and von zur Muehlen ([2001b], [2004]), and Onatski and Stock [2002].

10 Work in this field was initiated by Doyle [1982] and further developed in Dahleh and Diaz-Bobillo [1998] and Zhou et al. [1996], among others. Recent applications of this strand of robust control to monetary policy can be found in Onatski and Stock [2002], Onatski [2003], and Tetlow and von zur Muehlen [2001b].
for the model to attain a unique saddle-point stationary equilibrium.

2. Given the structural model, formulate the central bank’s depiction of the PLM used by agents to learn the structural model. Substitute this into the structural model to arrive at the actual law of motion. We refer to this as the reference model. While the reference model is the central bank’s best guess of the ALM, the bank understands that the reference model is only an approximation of the true ALM, and retains doubts about its local accuracy.\(^\text{11}\)

3. Specify a set of perturbations to the reference model structured in such a way as to isolate the possible misspecifications to which the reference model is regarded to be most vulnerable.

4. For a given policy rule, use \textit{structured singular value analysis} to determine the maximum allowable perturbations to the ALM that will bring the economy up to, but not beyond, the point of E-instability.

5. Finally, compute the policy for which the allowable range of misspecifications is the largest.

To begin, we rewrite the ALM from (2) and vectorize the disturbance, \(\epsilon_t\), to emphasize the stochastic nature of the estimating problem faced by agents,

\[
Y_{t+1} = \Pi Y_t + \tilde{\epsilon}_t,
\]

where \(Y_t = [1, y_{t-1}, v_t]'\) is of dimension \(n + 1\), \(\tilde{\epsilon}_t = [0\ 0\ \epsilon_{t+1}]'\) and

\[
\Pi = \begin{bmatrix}
1 & 0 & 0 \\
A + M(I + b)a & (N + Mb^2) & M(bc + cp) + P \\
0 & 0 & \rho
\end{bmatrix}.
\]

Notice that by using the ALM, we are modeling the problem from the policy authority’s point of view. The authority is taken as knowing, up to the perturbations we are about to add to the model, the structure of private agents’ learning problems.

\(^{11}\) It is sometimes argued that robust control—by which people mean minmax approaches to model uncertainty—is unreasonable on the face of things. The argument is that the worst-case assumption is too extreme, that to quote a common phrase, “if I worried about the worst case outcome every day, I wouldn’t get out of bed in the morning”. Such remarks miss the point that the worst-case outcome should be thought of as local in nature. Decision makers are envisioned as wanting to protect against uncertainties that are empirically indistinguishable from the data generating process underlying their reference models.
Potential misspecifications in parameter estimation are then represented by a perturbation block, \( \Delta \). In principle, the \( \Delta \) operator can be structured to implement a variety of misspecifications; robust control theory is remarkably rich in how it allows one to consider omitted lag dynamics, inappropriate exogeneity restrictions, missing nonlinearities, and time variation. This being the first paper of its kind, we keep our goals modest: in the language of linear operator theory, we will confine our analysis to linear time-invariant scalar (LTI-scalar) perturbations. LTI-scalar perturbations represent such events as one-time shifts and structural breaks in model parameters, as agents perceive them. Such perturbations have been the subject of study of parametric model uncertainty; see, e.g., Bullard and Eusepi [2005]. With this restriction, the perturbed model becomes:\footnote{Multiplicative errors in specification would be modeled in a manner analogous to (4): \( \epsilon_t = [A(1 - W_1\Delta W_2)]Y_t \).}

\[
\tilde{\epsilon}_t = Y_{t+1} - [\Pi + W_1\Delta W_2]Y_t, \\
= [Q - W_1\Delta W_2]Y_t,
\]

where \( Q = I_nL^{-1} - \Pi \), \( L \) is the lag operator, \( \Delta \) is a \( k \times k \) linear, time-invariant block-diagonal operator representing potentially destabilizing learning errors, and \( W_1 \) and \( W_2 \) are, respectively, \((n + 1) \times k \) and \( k \times (n + 1) \) selector matrices of zeros and ones that select which parameters in which equations are deemed to be subject to such errors. Either \( W_1 \) or \( W_2 \) can, in addition, be chosen to attach scalar weights to the individual perturbations so as to reflect relative uncertainties with which model estimates are to be regarded. The second line in (4) is convenient for analyzing stability of the perturbed model under potentially destabilizing learning errors.

The purpose of this paper is to find out how large, in a sense to be defined presently, the misspecifications represented by the perturbations in (4)—called the radius of allowable perturbations—can become without eliciting a failure of convergence to REE. A policy that expands the set of affordable perturbations is one that allows more room for misspecifications and thus offers improved prospects that policy will not be destabilizing.

Let \( D \) denote the class of allowable perturbations to the set of parameters of a model defined as those that carry with them the structure information of the perturbations. We define the radius, \( r > 0 \), to be some finite scalar and define \( D_r \) as the set of perturbations \( D \) modulated in size by this quantity.
in (4) that obey $||\Delta|| < r$, where $||\Delta||$ is the *induced norm* of $\Delta$ considered as an operator acting in a normed space of random processes. The scalar, $r$, can be considered a single measure of the maximum size of errors. A policy authority wishing to operate with as much room to maneuver as possible will act to maximize this range. For the tools to be employed here, norms will be defined in complex space.\(^{13}\) In what follows, much use is made of the concept of *maximum singular value*, conventionally denoted by $\sigma$.\(^{14}\) For reasons that will become clearer below, the norm of $\Delta$ that we shall use will be the $L_\infty$ norm of the function $\Delta(e^{i\omega})$, defined as the largest singular value of $\Delta(e^{i\omega})$ on the frequency range $\omega \in [-\pi, \pi]$:

$$||\Delta||_\infty = \left\{ \sup_\omega \max \text{eig} [\Delta'(e^{-i\omega})\Delta(e^{i\omega})] \right\}^{1/2},$$

(5)

where $\max \text{eig}$ denotes the maximum eigenvalue. The choice of $||\Delta||_\infty$ as a measure of the size of perturbations conveys a sense that the authority is concerned with worst-case outcomes.

Imagine two artificial vectors, $h_t = [h_{1t}, h_{2t}, ..., h_{kt}]'$ and $p_t = [p_{1t}, p_{2t}, ..., p_{kt}]'$, connected to each other and to $Y_t$ via\(^{15}\)

\[
\begin{align*}
p_t &= W_2 Y_t \\
h_t &= \Delta \cdot p_t.
\end{align*}
\]

Then we may recast the perturbed system (4) as the *augmented feedback loop*\(^{16}\)

\[
\begin{bmatrix}
Y_{t+1} \\
p_t
\end{bmatrix}
= \begin{bmatrix}
\Pi & W_1 \\
W_2 & 0
\end{bmatrix}
\begin{bmatrix}
Y_t \\
h_t
\end{bmatrix},
\]

(6)

\[
h_t = \Delta \cdot p_t.
\]

(7)

A reduced-form representation of this loop (from $h_t$ to $Y_t$ and $p_t$) is the transfer function

\[
\begin{bmatrix}
Y_t \\
p_t
\end{bmatrix}
= \begin{bmatrix}
G_1 \\
G_2
\end{bmatrix}
\begin{bmatrix}
h_t
\end{bmatrix},
\]

(8)

\(^{13}\) Our understanding is that the small, nascent literature on small gain theorems in the time domain has not reached the point they can be applied to macroeconomics. In particular, it is difficult to see how those methods can be scaled to behave well in multivariate economies. For this reason, we stick with the more well-understood tools of the frequency domain.

\(^{14}\) As is apparent from the expression in (5), the largest singular value, $\sigma(X)$, of a matrix, $X$, is the largest eigenvalue of $X'X$.

\(^{15}\) See Dahleh and Bobillo [1995], chapter 10.

\(^{16}\) Because the random errors in this model play no role in what follows, we leave out the $\epsilon$ vector.
where $G_1 = (I_nL^{-1} - \Pi)^{-1}W_1$, and $G_2 = W_2(I_nL^{-1} - \Pi)^{-1}W_1$ is a $n \times k$ matrix, where $k$ is the number of diagonal elements in $\Delta$. As we shall see, the stability of the interconnection between $h_t$ and $p_t$, representing a feedforward $p_t = G_2h_t$ and a feedback $h_t = \Delta \cdot p_t$, is critical. Note first that, together, these two relationships imply the homogenous matrix equation

$$0 = (I_k - G_2\Delta)p_t. \quad (9)$$

An E-stable ALM is also dynamically stable, meaning that $\Pi$ has all its eigenvalues inside the unit circle. To make this link, the following theorem is critical.

**Theorem 1**  *The Small Gain Theorem.*

Let $\text{Re}(s)$ denote the real part of $s \in \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, and let $\mathcal{H}_\infty$ denote the set of $\mathcal{L}_\infty$ functions analytic in $\text{Re}(s) > 0$. Furthermore, let $\mathcal{RH}_\infty$ designate the set of real, rational values in the $\mathcal{H}_\infty$-normed space. Suppose $G_2 \in \mathcal{RH}_\infty$ and $r > 0$. Then the interconnected system in (6)-(7) is well posed and internally stable for all $\Delta(s) \in \mathcal{RH}_\infty$ with

(a) $||\Delta|| \leq 1/r$, if and only if $||G_2(s)||_\infty < r$,

(b) $||\Delta|| < 1/r$, if and only if $||G_2(s)||_\infty \leq r$.


By assumption, $Q$, defined in (4), is invertible on the unit circle, allowing us to write\(^{17}\)

$$\det(Q)\det(I_k - G_2\Delta) = \det(Q)\det(I_k - W_2Q^{-1}W_1\Delta) = \det(Q)\det(I_k - Q^{-1}W_1\Delta W_2) = \det(Q - W_1\Delta W_2).$$

The preceding expressions establish the link between stability of the interconnection, $G_2$, and stability of the perturbed model: if $\det(I_k - G_2\Delta) = 0$, then the perturbed model (4) is no longer invertible on the unit circle, hence unstable, and vice versa. Thus, any policy rule that stabilizes $G_2$ also stabilizes the augmented system (6)-(7). The question to be asked

\(^{17}\) The small gain theorem links the stability of the loop between $p$ and $h$ under perturbations to the full system subject to model uncertainty. For some sufficiently large number $r$, such that $||\Delta||_\infty < r$, the determinant $\det(I_k - G_2\Delta) \neq 0$. Now raise $r$ to some value $r'$ such that $\det(I_k - G_2\Delta) = \det(I_k - W_2Q^{-1}W_1\Delta) = 0$. 

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then is how large, in the sense of \( \| \cdot \|_\infty \), can \( \Delta \) become without destabilizing the feedback system (6)-(7).

The settings we consider involve linear time-invariant perturbations, where the object is to find the minimum of the largest singular value of the matrix, \( \Delta \), from the class of \( D_r \) such that \( I - G_2\Delta \) is not invertible. The inverse of this minimum, expressed in the frequency domain, is the structured singular value of \( G_2 \) with respect to \( D_r \), defined at each frequency, \( \omega \in [-\pi, \pi] \),

\[
\mu[G_2(e^{i\omega})] = \min \left\{ \sigma[\Delta(e^{i\omega})] : \Delta \in D_r, \det(I - G_2\Delta)(e^{i\omega}) = 0 \right\},
\]

with the provision that if there is no \( \Delta \) such that \( \det(I - G_2\Delta)(e^{i\omega}) = 0 \), then \( \mu[G_2(e^{i\omega})] = 0 \).

The small gain theorem then tells us that, for some \( r > 0 \), the loop (6)-(7) is well posed and internally stable for all \( \Delta(.) \in D_r \) with \( \| \Delta \| < r \), if and only if \( \sup_{\omega \in \mathbb{R}} \mu[G_2(e^{i\omega})] \leq 1/r \).

Let \( \phi \) denote a vector of policy parameters. We can now formally state the problem that interests us as seeking a best \( \phi = \phi^* \) by finding a maximum value of \( \mu = \overline{\mu} \), satisfying

\[
\overline{\mu}(\phi^*) = \inf_{\phi} \sup_{\omega \in \mathbb{R}} \mu[G_2(e^{i\omega})]
\]

subject to the satisfaction of the saddle-point stability condition for the relevant model.

The solution to this problem is not amenable to analytical methods, except in special cases, an example of which we explore in the next section. Instead, we will employ efficient numerical techniques to find the lower bound on the structured singular value. The minimum of \( \mu^{-1}(G_2) \) over \( \omega \in [0, \pi] \) is exactly the maximal allowable range of misspecification for a given policy. A monetary authority wishing to give agents the widest latitude for learning errors that nevertheless allow the system to converge on REE selects those parameters in its policy rule that yield largest value of \( r \).

Figure 1 below provides a schematic representation of what is done. Given a policy rule, the ALM (and hence the reference model) is represented by the transition matrix, \( \Pi \). By assumption it is in the stable and determinate region of the space. The central bank chooses perturbation, \( \Delta \), to \( \Pi \). The largest feasible perturbation, \( \Pi_\Delta^* \)–the one that renders the largest radius, \( r \), defines a region shown by the ellipse within which any ALM, including but not restricted to the reference model ALM, will converge on a rational expectations
equilibrium. The weights, $W_1$ and $W_2$, determine the shape of the ellipse. Perturbations larger (in norm) than $r$ will push the ellipse over the line into indeterminate or explosive regions; smaller perturbations provide less robustness than the optimal one.

## 3 Two examples

We study two sample economies, one the very simple model of money demand in hyperinflations of Cagan [1956], the other the linearized neo-Keynesian model originated by Woodford ([1999], [2003]) among others. Closed-form solutions for $\mu$, being non-linear functions of the eigenvalues of models, are not generally feasible. However, some insight is possible through considering simple scalar example economies like the Cagan model. The second has the virtue of having been studied extensively in the literature on monetary policy design. It thus provides some solid benchmarks for comparison.
3.1 A simple univariate example

Consider a version of Cagan’s monetary model, cited in Evans and Honkapohja [2001], although our rendition differs slightly. The model has two equations, one determining (the log of) the price level, $p_t$, and the other a simple monetary feedback rule determining the (log of the) money supply, $m_t$:

$$m_t - p_t = -\kappa(E_t p_{t+1} - p_t)$$

$$m_t = \chi - \phi p_{t-1}.$$  

Normally, all parameters should be greater than zero; for $\kappa$ this means that money demand is inversely related to expected inflation. We will relax this assumption a bit later. Combining the two equations leads to:

$$p_t = \alpha + \beta E_t p_{t+1} - \gamma p_{t-1},$$  \hspace{1cm} (11)

where $\alpha = \chi/(1 + \kappa)$, $\beta = \kappa/(1 + \kappa)$, and $\gamma = \phi/(1 + \kappa)$. We assume for simplicity, $\beta, \gamma \neq 0$ to avoid degenerate models, and $\beta - \gamma \neq 1, \kappa \neq 0$ and $\kappa \neq -1$. Equation (12) below is the key to establishing the existence and uniqueness of saddle-point equilibrium

$$\beta \lambda_i^2 - \lambda_i - \gamma = \frac{\kappa}{1 + \kappa} \lambda_i^2 - \lambda_i - \frac{\phi}{1 + \kappa} = 0$$  \hspace{1cm} (12)

where $\lambda_i, i = 1, 2$ are the eigenvalues of the quadratic equations associated with (11). We require exactly one eigenvalue greater than unity for a saddle-point stable (equivalently, determinate, regular or unique) equilibrium to exist. If both eigenvalues are greater than unity the model is explosive; if neither is greater than unity the model is indeterminate (equivalently irregular, or nonunique). Inspection of the equation to the right of the first equality in equation (12) shows that the determinacy of the model is governed by the interaction between the structural money-demand parameter, $\kappa$, and the policy feedback parameter, $\phi$.

**Proposition 1** Assume that $\kappa \neq -1$. For $\kappa > -1$, determinacy requires: $\phi > -1$ and $\phi < 1 + 2\kappa$. For $\kappa < -1$, determinacy requires $\phi < -1$ and $\phi > 1 + 2\kappa$.

*Proof:* Equation (12) means that $| \beta - \gamma | = | (\kappa - \phi)/(1 + \kappa) | < 1$ is the condition for exactly one eigenvalue to be above unity. There are two cases. Assume first that $\kappa > -1$, which means that the denominator is always positive. Then simple arithmetic shows that $\{ \phi, \kappa \} \in \{ \mathcal{R} : \frac{\kappa - \phi}{1 + \kappa} < 1 \} = \{ \phi > -1 \} \cup \{ \phi < 1 + 2\kappa \}$ is implied. Now consider $\kappa < -1$. In this instance, $\{ \phi, \kappa \} \in \{ \mathcal{R} : \frac{\kappa - \phi}{1 + \kappa} < 1 \} = \{ \phi < -1 \} \cup \{ \phi > 1 + 2\kappa \}$.

\[ \square \]
Now let us assume that agents form expectations employing adaptive learning, and designate expectations formation in this way with the operator, $E^*_t$. The perceived law of motion for this model is assumed to be $p_t = a + b p_{t-1}$—the minimum state variable (MSV) solution—implying $E^*_t p_{t+1} = (1 + b) a + b^2 p_{t-1}$. The actual law of motion is found by substituting the PLM into the structural model:

$$p_t = [\alpha + \beta a(1 + b)] + (\beta b^2 - \gamma) p_{t-1}. \quad (13)$$

Following the steps outlined earlier, the ALM is $p_t = T_a(a, b) + T_b(a, b) p_{t-1}$ where $T$ defines the mapping $(\begin{array}{c} a \\ b \end{array}) = T(\begin{array}{c} a \\ b \end{array})$. This is

$$T_a: \quad a = \alpha + \beta a(1 + b) \quad (14)$$

$$T_b: \quad b = \beta b^2 - \gamma. \quad (15)$$

It is equation (15) that is key for the learnability of the model. But notice that (15) is identical to equation (12) with $b = \lambda_i$. This means that the conditions for learnability and determinacy are tightly connected in this particular model, as we shall discuss in detail below. The solutions to equations (14) and (15) are:

$$a = \alpha/[1 - \beta(1 + b)] \quad (16)$$

$$b = .5[1 \pm \sqrt{1 + 4\beta \gamma}]/\beta. \quad (17)$$

Equation (17) is quadratic with one root greater than or equal to unity, and the other less than unity. Designate the larger of the two values for $b$ as $b^+$ and the smaller as $b^-$. Existence of the REE requires us to choose the smaller root; otherwise, $b^+ \geq b^- > 1$. The ordinary differential equation system implied by this mapping is

$$\frac{d(\begin{array}{c} a \\ b \end{array})}{d\tau} = T(\begin{array}{c} a \\ b \end{array}) - (\begin{array}{c} a \\ b \end{array}),$$

for which the associated $DT$ matrix is derived by differentiating $[ T_a \quad T_b ]^t$ with respect to $a$ and $b$: 

$$DT = \begin{bmatrix} \beta(1 + b) & a \beta \\ 0 & 2\beta b \end{bmatrix}.$$
The eigenvalues of $DT - I$ are,

$$
\psi_1 = 2\beta b - 1 \quad (18)
$$

$$
\psi_2 = \beta(1 + b) - 1. \quad (19)
$$

Satisfaction of the weak E-stability condition requires that both eigenvalues be negative.

**Proposition 2** Determinate solutions of the Cagan model that are real are also E-stable.

**Proof:** Substitute the expression for $b^{-}$ into (18) to get $\psi_1 = -[1 + 4\phi/(1 + \kappa)^2]^{1/2}$ and do the same for (19) to arrive at $\psi_2 = \kappa/(1 + \kappa) - \frac{1}{2} + \psi_1$. Substitute $\psi_1$ into $b^{-}$ to arrive at: $b^{-} = \frac{1 + \kappa}{2\kappa}[1 + \psi_1]$. A necessary and sufficient condition for $\psi_1 < 0$, $\psi_1 \in \mathbb{R}$ is (P1): $\phi > (1 + \kappa)^2/(4\kappa) \equiv S$. Proposition 1 imposes restrictions on $\phi$ as a function of $\kappa$ to ensure determinacy. For $\kappa > -1$, these are $\phi > -1$ and $\phi < 1 + 2\kappa$. Substituting these into the expression for $\psi_1$ gives $\psi_1 < -[(1 + 3\kappa)/(1 + \kappa)]^2$ which readily yields $\psi_1 < 0$ and is real whenever $|b^{-}| < 1$ provided the solution is real. Simple substitution of $\psi_1$ into $\psi_2$ shows that $\psi_2$ is also negative. For $\kappa < -1$, a similar proof applies. □

Having outlined the connection between $b$ and $\beta$ (or $\kappa$) for E-stability, let us now consider unstructured perturbations to the ALM. Let $X_t = [1 \quad p_t]'$. The reference ALM model is then written as $X_t = \Pi X_{t-1}$, where

$$
\Pi = \begin{bmatrix} 1 & 0 \\ \alpha + \beta a (1 + b) & \beta b^2 - \gamma \end{bmatrix}
$$

is the model’s transition matrix. For simplicity, let us focus on $b$ as the object of concern to policy makers, and let the policy maker apply structured perturbations to $\Pi$, scaled by the parameter, $\sigma_b$. The scaling parameter can be thought of as a standard deviation, but need not be. Letting $W_1 = [0 \quad \sigma_b]'$ and $W_2 = [0 \quad 1]$, write the perturbed matrix $\Pi$ as:

$$
\Pi_{\Delta} = \begin{bmatrix} 1 & 0 \\ \alpha + \beta a (1 + b) & \beta b^2 - \gamma + \Delta \end{bmatrix}
$$

As in (8), the relevant matrices are defined in complex space. Accordingly, let $z = e^{i\omega}$, $\omega \in [-\pi, \pi]$. To find the maximal allowable perturbation, write

$$
G = \begin{bmatrix} z^{-1} - 1 & 0 \\ -\alpha - \beta a (1 + b) & z^{-1} - \beta b^2 + \gamma \end{bmatrix} = I \cdot z^{-1} - \Pi,
$$

which, defining $W_1 = [0 \quad \sigma_b]'$ and $W_2 = [0 \quad 1]$, is used to form $G_2$:

$$
G_2 = W_2 G^{-1} W_1
$$

$$
= \begin{bmatrix} 0 & 1 \\ \sigma_b z & \frac{z}{1 - (\beta b^2 - \gamma)z} \end{bmatrix}
$$

$$
= \begin{bmatrix} z \quad 0 \\ \sigma_b z \\ 1 - (\beta b^2 - \gamma)z \end{bmatrix}.
$$
It is for this expression that we seek the smallest structured singular value, $\mu$, as indicated by (10).

In the multivariate case, the scaling parameter $\sigma_b$, can be parameterized as the standard deviation of $b$ relative to $a$, although other methods of parameterization can be entertained. Doing so would reflect a concern for robustness of the decision maker and thus could also be thought of as a taste parameter. Since it is a relative term, it will turn out to be irrelevant in this scalar case, and so from here we set it to unity without loss of generality. The structured norm of $G_2$—equal to the absolute value of this last expression—is $\mu$. It is also easily established that the maximum of $\mu$ over the frequency range $\{-\pi, \pi\}$, is $\overline{\mu} = ||\Delta||_{\infty} = |G_2|$ arises at frequency $\pi$. Also, since at frequency $\pi$, $z = -1$, it follows that $|G_2| = \overline{\mu} = \frac{1}{1+b}$, or equivalently, the allowable perturbation is:

\[
\Delta = \frac{1}{\overline{\mu}} = 1 + b \\
= 1 + \beta b^2 - \gamma \\
= 1 + \frac{1}{2\beta} \left[ 1 - \sqrt{1 + 4\beta \phi/(1 + \kappa)} \right],
\]

which depends inversely on the policy parameter, $\phi$.\(^{18}\) Note also that while we have derived this expression for $\Delta$ by applying perturbations to the ALM, we would have obtained exactly the same result by working with the PLM.

If equation $\Delta$ is the allowable perturbation, conditional on a given $\phi$, then we can define a $\phi^*$ as the policy maker’s optimal choice of $\phi$, where optimality is defined in the sense of choosing the largest possible perturbation to $b$—call it $\Delta^*$—such that the model will retain the property of E-stability. Let us call this the maximum allowable perturbation. It is the $\Delta^*$ and the associated $\phi^*$ that is at a boundary where $\Delta$ is just above $-1$:

\[
\phi^* = 1 + 2\kappa - \epsilon,
\]

where $\phi < 1 + 2\kappa$ maintains stable convergence toward a REE and $\epsilon$ is an arbitrarily small positive constant necessary to keep $b + \Delta$ off the unit circle. Note that this expression for $\phi^*$ indicates that the monetary authority will always respond more than one-for-one to

\(^{18}\) Note that at frequency $\pi$, $1 - G_2 \Delta = 1 - \frac{\sigma_b}{(1+b)} \frac{(1+b)}{\sigma_b} = 0$ as required by the definition of $\mu$. 

deviations in lagged prices from steady state, with the extent of that over-response being a positive function of the slope of the money demand function. Substituting these expressions back into our perturbed transition matrix, we obtain:

\[
\Pi^*_\Delta = \begin{bmatrix}
1 & 0 \\
\alpha - \beta a (1 + b) & \beta b^2 - \gamma + \Delta \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
\alpha - \beta a (1 + b) & 1 + 2(\beta b^2 - \gamma) \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
\alpha - \beta a (1 + b) & -1 + \eta \\
\end{bmatrix},
\]

where \( \eta \) is an arbitrarily small number, as determined by \( \epsilon \) in (21). The preceding confirms that the authority’s policy is resilient to a perturbation in the learning model that pushes the transition matrix is to the borderline of instability. In other words, setting a \( \phi \) that allows for the maximal stable misspecification of the learning model is one that permits convergence to the REE.

Figure 2 shows the regions of dynamic stability and learnability for the Cagan model as functions of the structural parameters: the absolute interest elasticity of money demand, \( \kappa \), and the monetary policy feedback parameter, \( \phi \). As noted in the legend to the right, the determinate regions of the structural model are the blue areas, to the northeast and southwest of the figure. Proposition 1 above, warns the monetary authority to stay out of the indeterminate regions, the sliver of purple toward the southeast of the chart, that is possible for some \( \kappa > 0 \) when \( \phi < -1 \), and the larger region of purple to the west. Also to be avoided are the orange regions of explosive solutions to the north and south.

The E-stable region is the large area between the two dashed lines. The first thing to note is that E-stability does not imply determinacy: convergence in learning on indeterminate equilibria in the area where both \(-1 < \kappa < -1/2 \) and \(-1 < \phi < 0 \), is possible, corroborating a point made by Evans and McGough [2005] in a different context. In addition, learnability of unstable equilibria is also possible as shown by the orange regions between the two dashed lines. Indeed, even if one were to accept \textit{a priori} that \( \kappa > 0 \), as Cagan assumed, there are unstable equilibria that are learnable. At the same time, the figure clearly shows what Proposition 2 noted: for this model, determinate models are always E-stable; the blue region is entirely within the area bordered by the two dashed lines. It follows that in the special
case of the Cagan model, robustifying learnability is equivalent to maximizing the basin of attraction for the rational expectations equilibrium of the model. The loci of robust policies, $\phi^*$, conditional of values of $\kappa$, is shown by the thick diagonal line running from the south west to north east of the chart, and marked $\phi^* = 1 + 2\kappa$. The line shows that contrary to what unguided intuition might suggest, the robust policy does not choose a rule that is in the middle of the blue determinate and E-stable region, but rather chooses a policy that might be quite close to the boundary of indeterminacy for the REE. Doing so increases the region of E-stable ALMs—something that cannot be seen in the chart—and thereby enhances the prospects for convergence on an REE.

### 3.2 The canonical New Keynesian model

We now turn to an analysis of the canonical New Keynesian business cycle model. Evans and Honkapohja [2002] study this model to explore issues of determinacy and learnability for several optimal commitment rules. Bullard and Mitra [2002] likewise use the Woodford model to examine determinacy and learnability of variants of the Taylor rule.
The behavior of the private sector is described by two equations. The aggregate demand (IS) equation is a log-linearized Euler equation derived from optimal consumer behavior,

\[ x_t = E_t^* x_{t+1} - \sigma [r_t - E_t^* \pi_{t+1} - r^n_t], \]

(23)

and the aggregate supply (AS) equation—indeed, the price setting rule for monopolistically competitive firms is,

\[ \pi_t = \kappa x_t + \beta E_t^* \pi_{t+1}, \]

(24)

where \( x \) is the log deviation of output from potential output, \( \pi \) is inflation, \( r \) is a short-term interest rate controlled by the central bank, and \( r^n \) is the natural interest rate. For the application of Bullard and Mitra’s [2007] (BM) example, we assume that \( r^n_t \) is driven by a first-order autoregressive process,

\[ r^n_t = \rho r^n_{t-1} + \epsilon_{r,t}, \]

(25)

\( 0 \leq |\rho_r| < 1 \), and \( \epsilon_{r,t} \sim iid(0, \sigma^2_{r,t}) \). This is essentially Woodford’s [1999] version of this model, which specifies that aggregate demand responds to the deviation of the real rate, \( r_t - E_t \pi_{t+1} \) from the natural rate, \( r^n_t \).

We need to close the model with an interest-rate feedback rule. We study three types of policy rules. In the first set of experiments described in Section 3.3, a central bank chooses an interest rate setting in each period as a reaction to observed events, such as inflation and the output gap, without explicitly attempting to improve some measure of welfare. Instead, the policy authority is mindful of the effect its policy has on the prospect of the economy reaching REE and designs its rule accordingly. Bullard and Mitra [2007] study such rules for their properties in promoting learnable equilibria. We take this analysis further by seeking to find policy rules that maximize learnability of agents’ models when policy influences the outcome.

The information protocol in these experiments is as follows. The central bank knows the structural model and has access to the data. Economic agents see the data, which change over time, and formulate the perceived law of motion, a RLS estimation of the reduced form of the model. The data are regenerated each period, subject to the authority having implemented its policy and agents’ having made investment and consumption decisions based on their expectations.
We assume that agents mistakenly specify a vector-autoregressive model in the endogenous and exogenous variables of the model. That means we assume the learning model to be overparameterized in comparison with the model implied by the MSV solution. The scaling factors used in $W_1$ to scale the perturbations to the PLM are the standard errors of the coefficients obtained from an initial run of a recursive least squares regression of such a VAR with data being updated by the true model, given an arbitrary but determinate parameterization of the policy rule being studied.\footnote{As noted earlier, an alternative approach would be to revise the scalings with each trial policy, given that the VAR would likely change with each parameterization of policy. We leave this for future research.}

### 3.3 Simple interest-rate feedback rules

This section describes two versions of the Taylor rule analyzed by Bullard and Mitra [2002]. The complete system comprises equations (23)-(26), and the exogenous variable, $r^n_t$. The policy instrument is the nominal interest rate, $r_t$. The first policy rule specifies that the interest rate responds to lagged inflation and the lagged output gap. In their paper, BM study the role of interest-rate inertia and so include a lagged interest rate term.

\[
    r_t = \phi_x \pi_{t-1} + \phi_x x_{t-1} + \phi_r r_{t-1}
\]

McCallum has advocated such a *lagged data* rule because of its implementability, given that contemporaneous data are generally not available in real time to policy makers.

Some research suggests that forward-looking rules perform well in theory (see, e.g., Evans and Honkapohja [2002]) as well as in actual economies, such as Germany, Japan, and the US (see Clarida, Gali, and Gertler [1998]). Accordingly, BM propose the rule

\[
    r_t = \phi_x E^*_t \pi_{t+1} + \phi_x E^*_t x_{t+1} + \phi_r r_{t-1}
\]

where, as before, the expectations operator $E^*$ carries an asterisk to indicate that expectations need not be rational.

Finally, the most popular rules of this class are contemporaneous data rules—Taylor rules extended to allow for policy persistence—of which the following is our choice:

\[
    r_t = \phi_x \pi_t + \phi_x x_t + \phi_r r_{t-1}
\]
3.4 Results

We adopt BM’s calibration for the New Keynesian model’s parameters, $\sigma = 1/.157$, $\kappa = .024$, $\beta = .99$, and $\rho = .35$, the same calibration as in Woodford [2003]. We also set $\sigma_r = 0.01$. For reference purposes, it is useful to compare our results against those of rules that are not parameterized with robust learnability in mind. To facilitate this, we employ a standard quadratic loss function:

$$L_t = \frac{1000}{2} \sum_{j=0}^{\infty} \beta^j (\pi_{t+j} - \pi^*)^2 + \lambda_x x_{t+j}^2 + \lambda_r (r_{t+j} - r^*)^2.$$  \hspace{1cm} \text{(29)}$$

Walsh [2004] shows that with the values $\lambda_x = .077$ and $\lambda_i = .027$, equation (29) is the quadratic approximation to the social welfare function of the model. Rules that are computed to maximize the prospect of convergence to REE under the greatest possible misspecification of the ALM model in the manner described above will be referred to as "robust" or "robust learnable" rules. A credible benchmark against which to compare these robust rules, are what we shall refer to as optimized rules. These are rules that minimize (29) subject to (23), (24), (25) and one of either (26), (28) or (27). Such rules can be optimized using a standard hill-climbing algorithm using methods well described in the appendix to Tetlow and von zur Muehlen [2001a] among other sources.

Let us consider the lagged-data rule first. BM find that the determinacy of a unique rational expectations equilibrium, as well as convergence toward that equilibrium when agents learn adaptively, is extremely sensitive to the policy parameters, $\phi_r$, $\phi_x$, and $\phi_\pi$. Without some degree of monetary policy inertia, $(\phi_r > 0)$, this model is determinate and learnable, with the above calibrations, only if the Taylor principle holds, $(\phi_\pi > 1)$, and the response to the output gap is modest, $(\phi_x \leq 0.5)$. Insufficient or excessive responsiveness to either inflation or the output gap can in some instances lead to explosive instability or indeterminacy. Through simulation, BM establish the regions for the parameters that lead to determinacy as well as E-stability.

Table 1 shows our results. The table is broken into three panels. The upper panel—the rows marked (1) to (3)—shows optimized rules. The second panel, contains some results for the generic Taylor rule. Finally, the third panel shows our robust learnable rules. The next-to-last column of the table gives a measure of the total uncertainty that the PLM can...
tolerate under the cited policy. It is a measure of the maximal allowable deviation embodied in $1/\mu$. The last column shows the loss as measured by (29).

Table 1: Standard and robust learnable rules

<table>
<thead>
<tr>
<th>row</th>
<th>$\phi_x$</th>
<th>$\phi_\pi$</th>
<th>$\phi_r$</th>
<th>radius$^1$</th>
<th>$L^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>optimized rules:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>lagged data rule</td>
<td>(1)</td>
<td>0.052</td>
<td>0.993</td>
<td>1.13</td>
<td>1.07</td>
</tr>
<tr>
<td>contemporaneous data rule</td>
<td>(2)</td>
<td>0.053</td>
<td>0.995</td>
<td>1.12</td>
<td>1.06</td>
</tr>
<tr>
<td>forecast-based rule</td>
<td>(3)</td>
<td>0.286</td>
<td>0.999</td>
<td>1.32</td>
<td>0.88</td>
</tr>
<tr>
<td>standard rules:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Taylor rule</td>
<td>(4)</td>
<td>0.500</td>
<td>1.500</td>
<td>0</td>
<td>0.85</td>
</tr>
<tr>
<td>robust learnable rules:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>lagged data rule</td>
<td>(5)</td>
<td>0.065</td>
<td>0.40</td>
<td>1.10</td>
<td>1.16</td>
</tr>
<tr>
<td>contemporaneous data rule</td>
<td>(6)</td>
<td>0.052</td>
<td>1.21</td>
<td>1.41</td>
<td>1.13</td>
</tr>
<tr>
<td>forecast-based rule</td>
<td>(7)</td>
<td>0.040</td>
<td>2.80</td>
<td>0.10</td>
<td>2.32</td>
</tr>
</tbody>
</table>

1. Magnitude of the largest allowable perturbation. $r = \|W_1\Delta W_2\|_\infty$
2. Asymptotic loss, calculated according to eq. (29) in REE under the reference model.

Let us concentrate initially on our optimized rules along with the Taylor rule to provide some context for the robust learnable rules. The lagged data rule, shown in row (1), and the contemporaneous data rule, (2), are essentially the same. They both feature very small feedback on the output gap, and strong responses to inflation. Moreover, they also feature funds rate persistence that amounts to a first-difference rule; that is, a rule where the dependent variable is $\Delta r$ rather than $r$. The forecast-based rule, in line (3), has much stronger feedback on the output gap, although proper interpretation of this requires noting that in equilibrium the expectation of future output gaps will always be smaller than actual gaps because of the absence of expected future shocks and the internalization of future policy in the formulation of that expectation. Thus, the response of the funds rate to the expected future gap will not be as large as the feedback coefficient alone might lead one to believe.

These three rules confirm the received wisdom of monetary control in New Keynesian models, to wit: strong feedback on inflation, comparatively little on output, and strong persistence in funds rate setting. These rules are chosen to minimize the loss shown in the right-hand column of the table; the losses for all three are very similar, at a little over 3.6.

The results for the Taylor rule demonstrate, indirectly, the oft-discussed advantages for monetary control of persistence in the setting of the funds rate. Without such persistence,

---

$^{20}$ For comparison of the trials with each other and also to give a sense of natural units related to the scalings we employed, the radius is calculated as the $H_\infty$ norm of the scaled perturbations to the PLM model: $\text{radius} = \|W_1\Delta W_2\|_\infty$. 

---

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the Taylor rule produces losses that are substantially higher than those of the optimized rules.

Now let us turn to the robust learnable rules in the bottom panel of the table, concentrating for the moment on the lagged data and contemporaneous data rules shown in lines (5) and (6). The first thing to note is that the results confirm the efficacy of persistence in instrument setting. The robust learnable rules are at least as persistent—if persistence greater than unity is a meaningful concept—as the optimized rules. At the same time, while persistence is evidently useful for learnability, our results do not point to the hyper-persistence result, \( \phi_r \gg 1 \), that BM hint at. To understand this outcome, it is important to realize that while our results are related to the BM results, there are conceptual differences. BM describe the range of policy-rule coefficients for which the model is learnable, taking as given the model. We are describing the range of policy coefficients that maximizes the range of models that are still learnable. So while large values for \( \phi_r \) are beneficial to learnability holding constant the model and its associated ALM, at some point, they come at a cost in terms of the perturbations that can be withstood in other dimensions.

Now let us look at the costs and benefits of these two rules in comparison with their optimized counterparts. We measure the benefits by comparing the radii of robustness from the column second from the right, for various rules. For the optimized, outcome-based rules, shown in the first two rows of the table, the radii are about 1.06 or so, while those of their robustified counterparts range from 1.13 to 1.16. Thus the improvement in robustness of learnability would appear to be moderate. Costs are inferred by comparing the losses shown in the right-hand column of the table. The results show that the cost of maximizing learnability measured in terms of foregone performance in the REE is very small. Evidently, learnability can be robustified, to some degree, without much of any concomitant loss in economic performance, at least in the canonical NKB model.

Before moving on to forecast-based rules, let us consider the classic Taylor rule shown in the fourth row. Recall that the Taylor rule has been advocated as a policy that is at least reasonably robust across a fairly wide range of models. Here, however, the radius associated with the Taylor rule is shown to be quite small at 0.85. At the same time, the performance of the rule in terms of loss is relatively weak. Thus, to the extent that we can take claims of
the robustness of the Taylor rule with its original parameterization as applying to the issue of learnability, the rule would appear to come up a bit short.

Now let us examine the results for the forecast-based policy shown in the seventh row. Here the prescribed robust learnable policy is much different from the optimized rule shown in line (3). The robust rule essentially removes the policy persistence that the optimized policy calls for. The policy performance in the rational expectations equilibrium of the forecast-based robustly learnable rule is somewhat worse than its optimized counterpart, but notice that the radius of learnability is nearly triple that of the optimized rule.

While the superiority in terms of robustness of an (almost) non-intertial forecast-based rule is superficially at odds with Bullard and Mitra, the result really should not be all that surprising. Forecast-based rules leverage heavily the rational expectations aspects of the model—even more so than the contemporaneous and lagged data rules since there are rational expectations in the model itself and in the policy rule—and there is risk in leverage. The learnability of the economy is highly susceptible to misspecification in this area. This is, of course, just a manifestation of the problem that Bernanke and Woodford [1997] and others have warned about. The superiority in terms of learnability of the robustly learnable forecast-based rule notwithstanding, the deterioration in steady-state performance of this rule, relative to the optimized forecast-based rule, at 22 percent, is non-trivial. A central bank considering these options would have to consider its preference for robustness together with the likely period of time that the economy would spend in transition to REE.

We can obtain a deeper understanding of the effects of a concern for robust learnability on policy design by examining the properties of different calibrations of policy rules for their effects on the allowable perturbations. The magnitude of perturbations that a given model can tolerate, conditional on a policy rule, is given by the radius. The radii for the rules shown in Table 1 are in the column second from the right. We can, however, provide a visualization of radii mapped against policy-rule coefficients and judge how policy affects robust learnability.

Figure 3 provides one such visualization: contour maps of radii against the output-gap feedback coefficient, $\phi_x$, and inflation feedback coefficient, $\phi_\pi$, in this case for the contemporaneous data rule. The third dimension of policy, the feedback on the lagged fed funds rate,
The colors of the chart index the radii of allowable perturbations for each rule, with the bar at the right-hand side showing the tolerance for misspecification. The area in deep blue, for example, represents policies with no tolerance for misspecification of the model or learning whatsoever, either because the rule fails to deliver E-stability in the first place, or because it is very fragile. The sizable region of deep blue in the upper panel shows the area that violates the Taylor principle. The right of the deep blue region—where $\phi_r > 1$—we enter regions of green, where there is modest tolerance for misspecification that allows learnability.

In general, with no interest-rate smoothing, there is little scope for misspecification. Now let us look at the case where $\phi_r = 1$ in the bottom panel. Now the region of deep blue is relegated to the very south-west of the chart, as is the region of green. To the north-east of those are expansive areas of higher tolerance for misspecification. Evidently, at least some measure of persistence in policy is useful for robustifying learnability. Notice how there is a deep burgundy sliver of fairly strong robustness in the north-east part of the panel.

Figure 4 continues the analysis for the contemporaneous data rule by showing contour charts for two more levels of $\phi_r$. The upper panel shows the value for the rule that allows the maximum allowable perturbation as shown in line (6) of the table. With $\phi_r = 1.41$ the burgundy region of highest robustness is at its largest and the policy rule shown in line (6) of the table is within that region. More generally, the area of significant robustness—the redder regions—are collectively quite large. Finally, we go to the bottom panel of the figure which shows the results for a relatively high level of $\phi_r$. What has happened is that the regions shown in the top panel have rotated down and to the right as $\phi_r$ has risen. The burgundy region is now gone, and the red regions command much less space. Thus, while policy persistence is good for learnability, in terms of robustness of that result to misspecification, one can go too far.

Figures 3 and 4 cover the case of the contemporaneous data rule. We turn now to forecast-based rules. The results here look quite different, but the underlying message is very much the same. As before, Figure 5 shows the results for low levels of persistence in policy setting. The upper panel shows the static forecast-based rule. The deep blue
Figure 3: Contours of radii for the NKB model, contemporaneous data rule, selected $\phi_r$. 
Figure 4: Contours of radii for the NKB model, contemporaneous data rules, selected $\phi_r$.
areas to the left of $\phi_r = 1$ are areas of indeterminacy, as they were in Figure 3. There are, however, numerous blue "potholes" elsewhere in the panel. These are areas where the learnable equilibrium is feasible, but fragile. Notice, however, that these blue regions border very closely to burgundy regions where the allowable perturbations exceed 2; that is, the allowable perturbations are very large. The bottom panel shows contours covering the policy persistence level that is optimal, as shown in line (7) of table 1. There are fewer potholes. The optimally robust policy is toward the top of this chart.

Finally, let us examine Figure 6. The top panel shows that a small increase in $\phi_r$ from 0.10 to 0.12, reduces the number of potholes to nearly zero. The radii shown in the rest of the chart remain high, but the optimal policy is not in this region.

The bottom panel of the chart shows the contours for a modest and conventional value of funds rate persistence, $\phi_r = 0.50$. The potholes have now completely disappeared, but the large red region is less robust than the burgundy regions in the previous charts. Not shown in these charts are still higher levels of persistence. These involve still lower levels of robustness, with radii for $\phi_r > 1$ associated with radii that are less than half the magnitude of the maximum allowable perturbation for this rule. Higher levels of persistence in policy setting are deleterious for robustification of model learnability in inflation-forecast based policy rules.

Of course these particular results are contingent on the relative weightings for perturbations, captured in $W_1$, and our selection is just one of many that could have been made. For the numerical experiments, the weightings were set equal to the standard deviations of the coefficients of a first-order VAR for the output gap, inflation, the interest rate, and the natural rate, estimated at the beginning of each experiment using recursive least squares.

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21 Since degree of robustness is a function of the model’s eigenvalues and those are non-linear functions of the parameters of the model and of the learning mechanism, it is not possible to identify the source of these potholes. That said, it isn’t necessary either. The idea behind the methods described in this paper is to avoid the pitfalls of nonparametric errors.

22 The presence of the "potholes" in the chart for $\phi_r = 0.10$, wherein the optimally robust rule is found, and their near-absence for the chart $\phi_r = 0.12$ points to another concept of robustness. We assume the monetary authority knows the structural model. As a result, the economy cannot accidentally fall into one of the potholes shown in the figure. A worthwhile extension would be to allow the authority to have doubts about the structural parameters of the model, in addition to the learning mechanism. However the current paper, being the first in this area, is already ambitious enough and so we leave the issue for future research.

23 We tested $\phi_r$ up to nearly 20. What we found is that the radii fell as $\phi_r$ rose for intermediate levels, and then rose slowly again for $\phi_r \gg 1$. However, for no level of $\phi_r$ could we find radii that came anywhere close to the maximum allowable perturbation shown in row (7) of the table.
Figure 5: Contours of radii for the NKB model, forecast based rule, selected $\phi_r$. 
Figure 6: Contours of radii for the NKB model, forecast-based rule, selected $\phi_r$. 
This approximates the private sector’s problem, and should give a rough idea of the relative uncertainties associated with the coefficients of the PLM. Whether using estimated standard deviations to scale the relative impact of Knightian model uncertainty on the learning mechanism is proper or desirable can be debated, of course. For now the salient point is that robustness of learning in the presence of model uncertainty is not the same thing as choosing the rule parameters for which the E-stable region of a given model is largest.

4 SG learning as a perturbation

As we noted above, the key to the E-stability principle is the link between the convergence of the vector ordinary differential equation in meta-time and the local convergence in real time of the associated recursive least squares algorithm. But there is no assurance that agents use RLS to learn. In this section, we revisit our example of the New Keynesian business cycle model and treat stochastic gradient (SG) learning as a perturbation (in the robust control sense of that term) to the ALM. What we are looking for here is the answer to the question, if a central bank were to choose policy with a concern for robust learnability because it feared that private agents might use something other than RLS to learn, and if in fact agents used SG learning, would the chosen rule allow convergence on the REE? The answer is not obvious because robust learnability is not designed to withstand any specific (parametric) alternative to RLS, but is instead designed to withstand nonspecific (nonparametric) perturbations around the reference model of learning. If the central bank could be assured that the only two possible models of learning were RLS and SG learning, it could choose a policy that is convergent for both and be done. But we have no such assurance.

Let us consider the NKB model once again, just as we described it in section 3.2 above, except we now allow for a mark-up shock in the expectational Phillips curve:

\[ \pi_t = \kappa x_t + \beta E_t^r \pi_{t+1} + v_t \quad \text{with} \quad v_t = \rho_v v_{t-1} + \epsilon_{r,t} \]

and \(0 \leq |\epsilon_i| < 1, \epsilon_{i,t} \sim iid(0, \sigma_i^2)\) and \(i = [r^n, v]\). The added shock is necessary because otherwise the model is SG-stable any time it is E-stable. We set \(\rho_r = \rho_v = 0.35\), and

\(^{24}\)This section was inspired by a comment from an anonymous referee to whom we extend our thanks.
\( \sigma_i = 0.01 \) for all \( i \). As before, and consistent with conventional practice in the literature, we treat the \( \epsilon_{i,t} \) as observable. The added shock is needed to produce a meaningful distinction between RLS-stable economies and SG-stable economies.

Let us introduce the notation \( z_t = [v_t, r^n_t, r_{t-1}]' \) and \( y_t = [x_t, \pi_t, r_t]' \) and \( \gamma_t = [a_t, b_t, c_t]' \) and designate \( P_t = (z_t'z_t)^{-1} \) which is often called the precision matrix, the RLS algorithm can be expressed as:

\[
P_t = P_{t-1} + \frac{1}{t} (z_{t-1}z_{t-1}' - P_{t-1}) \tag{31}
\]

\[
\gamma_t = \gamma_{t-1} + \frac{1}{t} P_{t-1}^{-1} z_{t-1}'[z_{t-1}'(y_t - \gamma_{t-1}z_{t-1})] \tag{32}
\]

This is the recursive (discrete-time, real-time) counterpart to the T-mapping presented in section 2 above. If the E-stability principle holds, then the system (31) and (32) will converge on REE, provided that the recursion is initiated somewhere in the neighborhood of the equilibrium.

The SG algorithm simply drops equation (31) and sets \( P = I_n \) where \( n \) is the number of variables in the recursion. The simplicity of the SG algorithm is one reason why some researchers, e.g., Bullard and Eusepi [2005] and Evans et al. [2005] cite it as a plausible (or at least interesting) model of learning. The fact that it eschews the precision matrix means that SG learning is not independent of the scale of the regressors; this is one reason why its’ region of stability is typically weakly smaller than that of RLS learning, a feature that serves our purposes here. We hasten to add, however, that this is only an example. There are other models of learning that would be worthy of study in this context, including the trigger-switch model of Marcet and Nicolini [2003] and the various models of heterogeneous beliefs, such as Honkapohja and Mitra [2006].

We computed determinacy, E-stability, and SG-conditions for each point in a large three-dimensional grid of values for the policy rule parameters, \( \phi = \{\phi_x, \phi_\pi, \phi_r\} \) for each of the lagged data, contemporaneous data, and forecast-based policy rules studied previously. Then, for each point that was determinate, we computed real-time RLS and SG-stability tests and mapped out the regions where some or all of these conditions held. Finally, we computed the radii of allowable perturbations for each rule parameterization.\(^{25}\)

\(^{25}\)Sixteen points of \( \phi_x \) between 0.0 and 0.6; 31 points of \( \phi_x \) between 0.0 and 3.0; and 10 points of \( \phi_r \) between
To provide a parsimonious, two-dimensional visual representation of the results, we restrict our attention to the contemporaneous rule, (28), with $\phi_r = 0.8$, although this is not restrictive: we obtain similar results for all the values $0 < \phi_r < 1.8$ that we considered. The results are shown in Figure 7 below.

![Graph showing stability of selected calibrations of contemporaneous policy rule in the NKB model](image)

Figure 7: Stability of selected calibrations of contemporaneous policy rule in the NKB model

The values of the feedback coefficient on the output gap, $\phi_x$, are shown on the vertical axis, while the coefficient on the inflation rate, $\phi_p$, is shown on the horizontal axis. The dark blue region to the far west and south of the figure shows the region where neither E-stability nor SG-stability holds (and, equivalently in this case, the model is neither RLS nor SG)

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0.0 and 1.8 were computed, or 4,930 points for each rule. Each took several hours to compute.
convergent in real time). In fact, these regions are ones where the model is indeterminate. The large green region, lying like a chevron across the left-hand side of the figure is the region where the model is both E- and SG-stable (and determinate). Finally, the large, light-blue triangle toward the south-east of the figure is the region where the model is E-stable but not SG-stable. The yellow lines are *iso-radius contours*; that is, lines linking policy rule parameterizations of similar levels of non-parametric robust stability. The contour with the highest radius is in the far north of the figure, toward the west, labeled with a value of 1.8. There are two things to be noted about the rules in this region. First, with $\phi_x = 0.8$, $0.5 \lesssim \phi_x \lesssim 0.6$, and $0.7 \lesssim \phi_\pi \lesssim 1$, the parameterizations of the robust rules are broadly sensible; a central bank following such a rule would not be doing anything terribly radical. Second, and more important, the robustly learnable rules are well away from the region where policies are E-stable but not SG-stable.

5 Concluding remarks

We have argued that model uncertainty is a serious issue in the design of monetary policy. On this score we are in good company. Many authors have argued that minimizing a loss function subject to a given model presumed to be known with certainty is no longer best practice for monetary authorities. Central bankers must also take model uncertainty and learning into account. Where this paper differs from its predecessors is that we unify uncertainty about the learning mechanism used by private agents with the steps the monetary authority can take to address the problem. In particular, we examine a central bank that designs monetary policy to maximize the set of possible worlds in which learning by ill-informed private agents can converge on the rational expectations equilibrium.

Being a part of the data generating process, central banks can play a role in facilitating (or frustrating) the process of learning the REE through the design of policy. This paper begins from the premise that best practice for a monetary authority using simple instrument rules is to parameterize the rule that provides good performance not just in the steady state—that is, when convergence to REE has been achieved—but also in out-of-equilibrium behavior of the system. We further argue that foremost among the considerations of what constitutes good out-of-equilibrium behavior of an uncertain system under learning should
be the prospects for converging on an REE.

In pursuit of this goal, this paper has married the literature on adaptive learning to that of structured robust control to examine what policy makers can do to facilitate learning. We have introduced some tools with which the questions that Bullard and Mitra [2007] are asking can be broadened and generalized.

More narrowly we have also found that the warnings of Bernanke and Woodford [1997] are well placed; inflation-forecast-based monetary policy rules do present dangers. We have also shown that the conclusion that Bullard and Mitra [2007] point to is not as general as one might initially suppose.

Looking ahead, we see new research directions that broaden the scope of robustness by addressing a wider range of uncertainties against which a policy maker may wish to protect. Evans and McGough [2007], for example, show how to compute Taylor-type rules that will converge on an REE in a variety of models, taking the learning mechanism as given, whereas we take the structural model as given and investigate the implications of misspecification of learning rules. A fusion of the two approaches would seem to be worth investigating. The work on learning in heterogeneous agent economies—e.g., Honkapohja and Mitra [2006]—suggests that extensions to worlds in which the central bank is learning simultaneously with private agents would be useful.

Appendix

Determinacy

This Appendix reviews the determinacy and E-stability conditions associated with linear (or linearized) models, under the assumption that agents use the minimum state variable form of the model as the basis for their learning. Begin with the following linear rational expectations model:

\[ y_t = A + ME_t y_{t+1} + Ny_{t-1} + P v_t, \]  
(A.1)

where \( y_t \) is a vector of \( n \) endogenous variables, including, possibly, policy instruments, and \( v_t \) comprises all \( m \) exogenous variables. Equation (1) is general in that both non-predicted variables, \( E_t y_{t+1} \), and predicted variables, \( y_{t-1} \), are represented. By defining auxiliary
variables, e.g., $y^j_t = y_{t+j}, j \neq 0$, arbitrarily long (finite) lead or lag lengths can also be accommodated. Finally, extensions to allow lagged expectations formation; e.g., $E_{t-1}y_t$, and exogenous variables are straightforward to incorporate with no significant changes in results. Next, define the prediction error for $y_{t+1}$, to be $\xi_{t+1} = y_{t+1} - E_t y_{t+1}$. Under rational expectations, $E_t \xi_{t+1} = 0$, a martingale difference sequence. Evans and Honkapohja [2001] show that for at least one rational expectations equilibrium to exist, the stochastic process, $y_t$, that solves (A.1), must also satisfy:

$$y_{t+1} = -M^{-1}A + M^{-1}y_t - M^{-1}Ny_{t-1} - M^{-1}Pv_t + \xi_{t+1}$$  \hspace{1cm} (A.2)

We can express (A.2) as a first-order system:

$$\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} -M^{-1}A \\ 0 \end{bmatrix} + \begin{bmatrix} M^{-1} \\ I_n \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} -M^{-1} \\ 0 \end{bmatrix} P v_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xi_{t+1}$$

or, rewriting:

$$Y_{t+1} = F + BY_t + C v_t + D \xi_{t+1},$$  \hspace{1cm} (A.3)

where $Y = [y_t, y_{t-1}]'$. The model is **determinate** if (A.3) satisfies the Blanchard-Kahn conditions for stability, namely, that the number of characteristic roots of the matrix $B$ of norm less than unity equal the number of predetermined variables (taking $y_t$ to be scalar, this is one), meaning there is just one martingale difference sequence, $\xi_{t+1}$, that will render (A.2) stationary. If there are fewer roots inside the unit circle than there are predetermined variables, the model is **explosive** meaning that there is no martingale difference sequence that will satisfy the system. And if there are more roots inside the unit circle than there are predetermined variables, the model is said to be **indeterminate**, and there are infinite numbers of martingale difference sequences that make (A.2) **saddle-point stable**. The roots of $B$ are determined by the solution to the characteristic equation: $\lambda^2 - M^{-1}\lambda + M^{-1}N = 0$.

**E-stability**

Consider the minimum state variable (MSV) representation, advanced by McCallum [1983]. Let us assume that $v_t$ is observable and follows a first-order stochastic process, $v_t = \rho v_{t-1} + \epsilon_t$ where $\epsilon_t$ is an iid white noise process. Assume the $\rho$ matrix is diagonal. Then we can write
the following *perceived law of motion* (PLM):

\[ y_t = a + by_{t-1} + cv_t \equiv \Pi^p z_t. \]  \hspace{1cm} (A.4)

Rewrite equation (A.1) slightly, and designate expectations formed using adaptive learning with a superscripted asterisk on the expectations operator, \( E_t^* \):

\[ y_t = A + ME_t^* y_{t+1} + Ny_{t-1} + Pv_t. \]  \hspace{1cm} (A.5)

Then, leading (A.4) one period, taking expectations, substituting (A.4) into the result, and finally into (A.5), we obtain the *actual law of motion*, (ALM), the model under the influence of the PLM:

\[ y_t = A + M(I + b)a + (N + Mb^2)y_{t-1} + (M(bc + c\rho) + P)v_t. \]  \hspace{1cm} (A.6)

So the MSV solution will satisfy the mapping from PLM to ALM:

\[
\begin{align*}
A + M(I + b)a &= a, \\
N + Mb^2 &= b, \\
M(bc + c\rho) + P &= c.
\end{align*}
\]

E-stability depends then on the mapping of the PLM on to the ALM, defined from (A.6):

\[ T(a, b, c) = [A + M(I + b)a, N + Mb^2, M(bc + c\rho) + P] \]  \hspace{1cm} (A.7)

The fixed point of this mapping is a MSV representation of a REE, and its convergence is given by the matrix differential equation:

\[ \frac{d}{dT}(a, b, c) = T(a, b, c) - (a, b, c). \]  \hspace{1cm} (A.8)

Convergence is assured if certain eigenvalue conditions for the following matrix differential equations are satisfied.

\[
\begin{align*}
\frac{da}{dT} &= [A + M(I + b)]a - a, \\
\frac{db}{dT} &= Mb^2 + N - b, \\
\frac{dc}{dT} &= M(bc + c\rho) + P - c.
\end{align*}
\]  

38
As shown by Evans and Honkapohja [2001], the necessary and sufficient conditions for E-stability are that the eigenvalues of the following matrices have negative real parts:

\[
DT_a - I = M(I + b) - I,
\]
\[
DT_b - I = b' \otimes M + I \otimes Mb - I,
\]
\[
DT_c - I = \rho' \otimes M + I \otimes Mb - I.
\]

The important points to take from equations (A.9) are that the conditions are generally multivariate in nature—meaning that the coefficients constraining the intercept term, \(a\), can be conflated with those of the slope term, \(b\), and that the coefficients of both the PLM and the ALM come into play.

References


